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Augmented Beam Elements Using Unit Deflection Shapes Together with a Finite Element Discretisation of the Cross Section

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Abstract

This thesis covers the field of augmented beam Finite Elements for geometrical and material linear, prismatic structures in the fields of static and dynamic excitations. An approach of superposed unit deflection shapes which are defined on a two dimensional Finite Element mesh of the cross section is used. This Finite Element mesh of the cross section is also used for the preparation of the unit deflection shapes by solving different differential equations such that unit deflection shapes emerge which most effectively cover the physical effects occurring on beam shaped structures with a three dimensional material law. The concept of unit deflection shapes is in analogy with the concept of warping torsion of beam elements. The method is validated and visualised with several numerical examples which are compared to the results of three dimensional standard Finite Element models.

Kurzfassung

Diese Arbeit behandelt erweiterte Finite Balkenelemente im materiell und geometrisch linearen Anwendungsbereich für prismatische Strukturen unter statischen und dynamischen Lasten. Dabei wird eine Superposition von Einheitsverformungszuständen eingesetzt, die auf einem zweidimensionalen Finite Elemente Netz des Querschnitts definiert sind. Dieses Finite Elemente Netz wird auch für die Erzeugung der Einheitsverformungszustände verwendet. Dabei werden auf diesem Netz verschiedene Differentialgleichungen gelöst, um Einheitsverformungen zu erhalten, mit denen das Verhalten einer balkenartigen Struktur mit dreidimensionalem Materialgesetz möglichst effektiv erfasst werden kann. Das Konzept der Einheitsverformungszustände steht dabei in Analogie zu dem der Wölbkrafttorsion. Die Methode wird in der Arbeit validiert und die Ergebnisse mit denen von dreidimensionalen Finite Elemente Berechnungen verglichen.

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List of Symbols

The choice of symbols and notation was made according to popular text books and lecture notes on the topic of Finite Elements and Beam Theory. Different denominations for the same quantity were avoided as far as possible. Quantities which are used only once are declared by figures or text at the position of their use.

Greek Letters

$\boldsymbol{\psi}$	–	Unit deflection shapes
$[\boldsymbol{\Phi}]$	–	Matrix of unit deflection shapes
$\boldsymbol{\Psi}$		Eigenvector for a homogeneous problem
$\underline{\psi}$	–	Generalised amplitude of the primary torsional warp deflection
ν	–	*Poisson*[1] ratio
ρ	$\frac{\mathrm{kg}}{\mathrm{m}^3}$	density
$\boldsymbol{\varepsilon}, [\varepsilon]$	–	strain tensor and matrix of strain components (see section 2.2)
$\underline{\varphi}_x, \underline{\varphi}_y, \underline{\varphi}_z$	rad	(Generalised) rigid body rotations of the cross section
$\gamma_{ij}, \boldsymbol{\gamma}$	–	shear strain component, vector of shear strains
λ	m	Wave length
$\boldsymbol{\sigma}, [\sigma]$	$\frac{\mathrm{N}}{\mathrm{m}^2}$	Stress tensor and matrix of stress components (see section 2.3)
ω	m	Warp deflection field
ω_V	m	Primary torsional warp shape
Ω	$\frac{\mathrm{rad}}{\mathrm{s}}$	Angular frequency

[1] Siméon Denis Poisson (∗1781, †1840)

Latin Letters

$[0]$	–	Zero matrix
A	m^2	Area
$[B]$		Matrix of derived shape functions (FEM)
c	$\frac{\mathrm{m}}{\mathrm{s}}$	Wave speed (propagation)
$[D], [D_\omega], [D_{Mn}], [D_{Ms}]$	$\frac{\mathrm{N}}{\mathrm{m}^2}$	Material matrix (see section 2.3)
E	$\frac{\mathrm{N}}{\mathrm{m}^2}$	*Young*'s modulus
EA	N	Longitudinal stiffness
EI_y, EI_z, EI_{yz}	Nm^2	Bending stiffness (for composite beams)
EI_ω	Nm^4	Warping resistance (in literature also $EA_{\omega\omega}$ or EC_M)
f	$\frac{1}{\mathrm{s}}$	Frequency
G	$\frac{\mathrm{N}}{\mathrm{m}^2}$	Shear modulus
$GA_{Sy}, GA_{Sz}, GA_{Syz}$	$\frac{\mathrm{Nm}^2}{\mathrm{m}^2}$	Shear stiffness (see appendix A.5.2)
$GA_{S\omega}, GA_{Sy\omega}, GA_{Sz\omega}$	$\frac{\mathrm{Nm}^2}{\mathrm{m}^2}$	Stiffness against warping torsion shear deformation (see appendix A.5.2)
GI_t	Nm^2	Torsion constant (see appendix A.5.1.1)
i	–	Imaginary unit ($i = \sqrt{-1}$)
$[I]$	–	Identity matrix
ρI_ω	$\mathrm{kg\ m}^3$	Dynamic warping inertia
ρI_ϑ	kg m	Dynamic rotational inertia
k_x	$\frac{\mathrm{rad}}{\mathrm{m}}$	Wave number
$[K]$		Stiffness matrix
$[M]$		Mass matrix
M_B	Nm^2	Bimoment
M_x	Nm	Torsional moment
M_{xV}	Nm	Primary torsional moment (*de St.-Venant* component)
M_{xS}	Nm	Secondary torsional moment (from warping torsion)
M_y, M_z	Nm	Bending moments
N	N	Normal force
$[N]$	–	Matrix of shape functions (FEM)
$p, \mathbf{p}, [p]$	N	Load, load vector, nodal load matrix
$q, \mathbf{q}, [q]$	N	General force, force vector, nodal force matrix
Q_y, Q_z	N	Transverse forces
t	s	Time
$u_x, u_y, u_z, \mathbf{u}, [u]$	m	Deflection components, deflection vector, nodal deflection matrix

$\underline{u}_x$,$\underline{u}_y$,$\underline{u}_z$	m	(Generalised) rigid body deflections of the cross section
$[U]$	–	Transmission matrix (see section 3.1.2)
V	m^3	Volume
W	Nm	Work
$\underline{v}$	–	Participation factor
$\breve{y}, \breve{z}$	m	Coordinates with reference to the centre of gravity $(y_S; z_S)$
$\acute{y}, \acute{z}$	m	Coordinates with reference to the shear centre/ centre of rotation $(y_M; z_M)$
x,y,z	m	(*Cartesian*) coordinates
y_M, z_M	m	Coordinates of the shear centre/ centre of rotation
y_S, z_S	m	Coordinates of the centre of stiffness
y_I, z_I	m	Coordinates of the centre of gravity/mass

Accents and Symbols

$\overset{\circ}{\square}$	Known value from a previous computation or state
$\underline{\square}$	Generalised quantity
$\bar{\square}$	Conjugate (complex) value
$\square_{,x}$	Partial derivative with respect to x
$[\square]$	Indicates a matrix
$[\square]^T$	Transpose of a matrix
$[\square]^{-1}$	Inverse of a matrix
$\dot{\square}$	Derivative with respect to time
$\ddot{\square}$	Second derivative with respect to time
$\hat{\square}$	Quantity in the *Fourier* domain
$\Re(\square)$	Real part of a complex quantity
$\Im(\square)$	Imaginary part of a complex quantity
$\boldsymbol{\tau}$	(bold face) Indicates a vector or tensor of higher order
$\delta\square$	Virtual quantity (principle of virtual work)
$f^{(n)}$	n^{th} Derivative of a function f

Abbreviations

BEM	Boundary Element Method
CAD	Computer aided design
DOF	Degree of freedom
FEM	Finite Element Method
FT	*Fourier*[2] transformation
GBT	Generalised beam theory
GEP	Generalised eigenvalue problem
IFT	Inverse *Fourier* transformation
ITM	Integral transformation method
QEP	Quadratic eigenvalue problem
SDOF	Single degree of freedom
SEA	Statistical energy analysis
SEP	Standard eigenvalue problem

[2] Jean Baptiste Joseph Fourier (∗1768, †1830)