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Shearlet Coorbit Spaces, Shearlet Transforms and Applications in Imaging

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Abstract

Directional multiscale representation of images to address curved singularities has received much attention in harmonic analysis in the last 25 years. In particular, shearlets and curvelets provide an optimally sparse approximation of cartoon-like images. Shearlets possess a uniform construction for both the continuous and the discrete setting. This and the underlying group structure let them gain attraction in various theoretical and applied fields. In this thesis we contribute to both the continuous and the discrete setting.

The outstanding property of the shearlet transform is the underlying shearlet group. Having this group we discover isomorphisms to other groups, namely extended Heisenberg groups and subgroups of the symplectic group. Interestingly, the connected shearlet group with positive dilations has an isomorphic symplectic subgroup while this is not true for the full shearlet group with all non-zero dilations. To this end we prove the general result that there exist, up to the adjoint action of the symplectic group, only one embedding of the extended Heisenberg algebra into the Lie algebra of the symplectic group. Besides the usual full and connected shearlet groups we also deal with Toeplitz shearlet groups.

Shearlet coorbit spaces are canonical smoothness spaces designed by applying the general coorbit theory of Feichtinger and Gröchenig. We examine structural properties of these shearlet coorbit spaces. In particular we focus on subspaces resembling shearlets on the cone in three dimensions. We show an embedding into Besov spaces. Further, we examine traces onto the coordinate planes and their embedding into lower dimensional Besov and shearlet coorbit spaces. The results are based on atomic decomposition of Besov spaces and molecules in shearlet coorbit spaces. We further establish Toeplitz shearlet coorbit spaces. Based on the group isomorphisms we prove that isomorphic groups with equivalent representations give rise to isomorphic coorbit spaces leading to the definition of metaplectic coorbit spaces.

In the discrete setting and its applications we describe the implementation details of a fast and fully finite shearlet transform based on the FFT. To this end we construct a discrete Parseval shearlet frame. We further describe how the discrete shearlet transform can be incorporated into convex imaging functionals for segmentation, decomposition and inpainting. We show that the shearlet regularized segmentation outperforms the TV-regularizer on curved textures. The method can also cope with regularizers based on non-local means.

We introduce a novel quadrature operator called linearized Riesz transform that corresponds to the shear operator. In contrast the usual Riesz operator covaries with orthogonal transforms and in particular with rotations. We prove properties of the new transform and analyze the performance compared to the usual Riesz transform numerically. Based on the linearized Riesz transform we introduce finite discrete quasi-monogenic shearlets. Numerical experiments show the alignment of the directional information obtained from the shearlets and the quasi-monogenic orientation. We apply the quasi-monogenic shearlet transform in the analysis of textures.

Zusammenfassung

Die Zerlegung von Bildern auf verschiedenen Skalen und und in verschiedene Richtungen mit Hinblick auf Singularitäten entlang von Kurven hat in den letzten 25 Jahren viel Aufmerksamkeit in der harmonischen Analysis erlangt. Insbesondere Shearlets und Curvelets bieten eine optimal dünne Approximation von cartoon-artigen Bildern. Shearlets werden im Kontinuierlichen und im Diskreten einheitlich konstruiert. Dies und die zugrunde liegenden Gruppenstruktur machen Shearlets in verschiedenen theoretischen und angewandten Bereichen interessant. In dieser Arbeit behandeln wir sowohl kontinuierliche als auch diskrete Shearlets.

Die Shearlet-Transformation zeichnet sich durch die zugrunde liegende Gruppenstruktur aus. Ausgehend von dieser Gruppe lassen sich Isomorphismen zu anderen bekannten Gruppen, den erweiterten Heisenberg-Gruppen und Untergruppen der symplektischen Gruppe, finden. Interessanterweise hat die zusammenhängende Shearlet-Gruppe mit positiven Dilatationen eine isomorphe symplektische Untergruppe. Dagegen existiert keine solche Untergruppe für die volle Shearlet-Gruppe mit allen Dilatationen ungleich null. Für den Beweis zeigen wir allgemein, dass es, bis auf das Adjungieren mit symplektischen Matrizen, nur eine Einbettung der erweiterten Heisenberg-Algebra in die Lie-Algebra der symplektischen Gruppe gibt. Neben den zusammenhängenden und vollen Shearlet-Gruppen behandeln wir auch Toeplitz-Shearlet-Gruppen.

Shearlet-Coorbit-Räume sind kanonische Glattheitsräume, die sich durch das Anwenden der allgemeinen Coorbit-Theorie von Feichtinger und Gröchenig konstruieren lassen. Wir untersuchen die Struktur der Shearlet-Coorbit-Räume. Insbesondere konzentrieren wir uns auf Unterräume in drei Dimensionen die Shearlets „on the cone“ nachbilden. Wir zeigen eine Einbettung in Besov-Räume und beweisen Spursätze für die Projektionen auf die Koordinatenebenen und deren Einbettung in niedriger-dimensionale Besov- und Shearlet-Coorbit-Räume. Für die Ergebnisse benötigen wir atomare Zerlegungen von Besov-Räumen und Moleküle in Shearlet-Coorbit-Räumen. Darüber hinaus führen wir Toeplitz-Shearlet-Coorbit Räume ein. Basierend auf den Gruppen-Isomorphismen beweisen wir, dass isomorphe Gruppen mit äquivalenten Darstellungen zu isomorphen Coorbit-Räumen führen. In diesem Zusammenhang betrachten wir auch metaplektische Coorbit-Räume.

Für die Anwendung der Shearlet-Transformation beschreiben wir die Details der Implementierung einer schnellen und endlichen Shearlet-Transformation, die auf der schnellen Fouriertransformation (FFT) basiert. Eine wichtige Grundlage ist die Konstruktion eines diskreten Parseval-Frames. Die diskrete Shearlet-Transformation kann als Regularisierer in konvexen Funktionalen der Bildverarbeitung verwendet werden, als Beispiele dienen Segmentierung, Dekomposition und Inpainting. Numerische Beispiele zeigen, dass die Shearlet-Regularisierung linienartige Texturen besser segmentieren kann als die TV-Regularisierung und vergleichbare Ergebnisse wie NL-means Regularisierer liefert.

Mit der linearisierten Riesz-Transformation führen wir einen neuen Quadratur-Operator ein, der mit der Scherung harmoniert. Die gewöhnliche Riesz-Transformation ist dagegen invariant unter orthogonalen Transformationen wie zum Beispiel Rotationen. Wir bewei-

sen Eigenschaften der neuen Transformation und vergleichen sie numerisch mit der gewöhnlichen Riesz-Transformation. Mit Hilfe der linearisierten Riesz-Transformation führen wir dann endliche diskrete quasi-monogene Shearlets ein. Die numerischen Experimente zeigen, dass die Richtung der Shearlets gut mit der quasi-monogenen Orientierung übereinstimmt. Als Beispiel für eine Anwendung werden Texturen analysiert.

Contents

List of Publications

Parts of this thesis have been published in the following articles and preprints. The numbers of the respective articles are the same as in the full bibliography at the end of the thesis.

[34] S. Dahlke, S. Häuser and G. Teschke. Coorbit space theory for the Toeplitz shearlet transform. *International Journal of Wavelets, Multiresolution and Information Processing*, 10(4), 2012.

[35] S. Dahlke, S. Häuser, G. Steidl and G. Teschke. Shearlet coorbit spaces: traces and embeddings in higher dimensions. *Monatshefte für Mathematik*, 169(1):15–32, 2013.

[36] S. Dahlke, F. De Mari, E. De Vito, S. Häuser, G. Steidl and G. Teschke. Different faces of the shearlet group. *arXiv Preprint*, 1404.4545, 2014.

[92] S. Häuser and J. Ma. Seismic data reconstruction via shearlet-regularized directional inpainting. *Preprint University of Kaiserslautern*, 2012.

[93] S. Häuser and G. Steidl. Convex multiclass segmentation with shearlet regularization. *International Journal of Computer Mathematics*, 90(1):62–81, 2013.

[94] S. Häuser and G. Steidl. Fast Finite Shearlet Transform: a tutorial. *arXiv Preprint*, 1202.1773, 2014.

[95] S. Häuser, B. Heise and G. Steidl. Linearized Riesz transform and quasi-monogenic shearlets. *International Journal of Wavelets, Multiresolution and Information Processing*, 12(3), 2014.

[98] B. Heise, S. E. Schausberger, S. Häuser, B. Plank, D. Salaberger, E. Leiss-Holzinger and D. Stifter. Full-field optical coherence microscopy with a sub-nanosecond supercontinuum light source for material research. *Optical Fiber Technology*, 18(5):403–410, 2012.

Notation and Symbols

Continuous and Discrete Fourier Transform

The *Fourier transform* $\mathcal{F}\colon L_2(\mathbb{R},\mathbb{C}) \to L_2(\mathbb{R},\mathbb{C})$ is given by

$$\mathcal{F}f(\omega) := \hat{f}(\omega) := \int_{\mathbb{R}^d} f(t)\,\mathrm{e}^{-2\pi\mathrm{i}\langle\omega,t\rangle}\,\mathrm{d}t\,.$$

We will frequently use that for a linear and invertible operator $A \in \mathbb{R}^{d\times d}$ the relation

$$\mathcal{F}(f(A^{-1}\cdot))(\omega) = |\det(A)|\hat{f}(A^{\mathrm{T}}\omega) \tag{I.1}$$

holds true. The inverse Fourier transform is given by

$$\mathcal{F}^{-1}F(t) = \check{F}(t) = \int_{\mathbb{R}^d} F(\omega)\,\mathrm{e}^{2\pi\mathrm{i}\langle\omega,t\rangle}\,\mathrm{d}\omega\,.$$

By *Plancherel's theorem* we know for $f, g \in L_2(\mathbb{R},\mathbb{C})$ that

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \quad \text{and in particular} \quad \|f\|_2 = \|\hat{f}\|_2. \tag{I.2}$$

For $M, N \in \mathbb{N}$ we consider the digital grid

$$\mathcal{G} := \{(m,n) : m = 0,\dots,M-1,\ n = 0,\dots N-1\}$$

and the set of discrete frequencies

$$\Omega := \left\{(\omega_1,\omega_2) : \omega_1 = -\left\lfloor\tfrac{M}{2}\right\rfloor,\dots,\left\lceil\tfrac{M}{2}\right\rceil - 1,\ \omega_2 = -\left\lfloor\tfrac{N}{2}\right\rfloor,\dots,\left\lceil\tfrac{N}{2}\right\rceil - 1\right\}.$$

For $f\colon \mathcal{G} \to \mathbb{R}$ the *discrete Fourier transform (DFT)* is defined by

$$\hat{f}(\omega) = \sum_{m\in\mathcal{G}} f(m)\,\mathrm{e}^{-2\pi\mathrm{i}\left\langle\omega,\binom{m/M}{n/N}\right\rangle} = \sum_{m\in\mathcal{G}} f(m)\,\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{\omega_1 m}{M}+\frac{\omega_2 n}{N}\right)}, \quad \omega \in \Omega$$

and the *inverse discrete Fourier transform (IDFT)* by

$$f(m) = \frac{1}{MN}\sum_{\omega\in\Omega} \hat{f}(\omega)\,\mathrm{e}^{2\pi\mathrm{i}\left\langle\omega,\binom{m/M}{n/N}\right\rangle} = \frac{1}{MN}\sum_{\omega\in\Omega} \hat{f}(\omega)\,\mathrm{e}^{2\pi\mathrm{i}\left(\frac{\omega_1 m}{M}+\frac{\omega_2 n}{N}\right)}, \quad m \in \mathcal{G}.$$

The discrete Fourier transform can be efficiently computed using the FFT. Useful is Parseval's formula for $f, g\colon \mathcal{G} \to \mathbb{R}$ which says similar to Plancherel's Theorem in the continuous setting that

$$\langle f, g\rangle = \frac{1}{MN}\langle \hat{f}, \hat{g}\rangle \quad \text{and also here} \quad \|f\|_F = \frac{1}{MN}\|\hat{f}\|_F. \tag{I.3}$$

The most important property is that the convolution of two functions (respectively vectors) can be computed as a product of their Fourier transforms. This holds both in the continuous and in the discrete setting.

More details about the Fourier transform and its properties can be found for example in [63].

Symbols

$\mathbb{N}$	Set of natural numbers without 0
$\mathbb{N}_0$	Set of natural numbers with 0, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^+$	Set of all positive real numbers without 0
$\mathbb{R}^*$	Set of real numbers without 0, $\mathbb{R} \setminus \{0\}$
$\mathbb{C}$	Set of all complex numbers
C^k	Space of functions $f\colon \mathbb{R} \to \mathbb{R}$ which are k-times continuously differentiable
$L_p(\Omega)$	Space of all measurable functions $f\colon \Omega \to \mathbb{C}$ such that the integral $\int_\Omega \lvert f(x)\rvert^p \,\mathrm{d}x$ is finite ($1 \le p < \infty$) and two functions are identified if they are equal almost everywhere
$L_\infty(\Omega)$	Space of all measurable functions $f\colon \Omega \to \mathbb{C}$ such that there exists a constant K with $\lvert f(x)\rvert \le K$ almost everywhere on Ω
$L_{p,w}(\Omega)$	Weighted L_p space, space of all measurable functions $f\colon \Omega \to \mathbb{C}$ such that $fw \in L_p(\Omega)$
$\mathcal{G}$	Digital grid
$\mathcal{H}$	Hilbert space
$\mathcal{I}$	Countable (possibly infinite) index set
$\lesssim$	Less or equal up to a constant independent of the involved parameters, i.e., $f \lesssim g \iff f \le C\, g$ with some generic constant $C \ge 0$
$\operatorname{atan2}(y,x)$	Arctangent function with two arguments $\operatorname{atan2}(y,x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{for } x > 0, \\ \arctan\left(\frac{y}{x}\right) + \pi & \text{for } y \ge 0, x < 0, \\ \arctan\left(\frac{y}{x}\right) - \pi & \text{for } y < 0, x < 0, \\ \frac{\pi}{2} & \text{for } y > 0, x = 0, \\ -\frac{\pi}{2} & \text{for } y < 0, x = 0, \\ 0 & \text{for } x = y = 0 \end{cases} = \begin{cases} \arccos\left(\frac{y}{\sqrt{x^2+y^2}}\right) & \text{for } y \ge 0, \\ -\arccos\left(\frac{y}{\sqrt{x^2+y^2}}\right) & \text{for } y < 0 \end{cases}$
∇	Gradient operator
$\frac{\partial}{\partial x_i}$	Partial derivative operator in x_i direction
$f * g$	Convolution of two functions f and g
ι_A	Indicator function $\iota_A\colon \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ of A defined by $\iota_A(x) := \begin{cases} 0 & \text{for } x \in A, \\ +\infty & \text{otherwise} \end{cases}$
χ_A	Characteristic function $\chi_A\colon \mathbb{R}^d \to \mathbb{R}$ of A defined by $\chi_A(x) := \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{otherwise} \end{cases}$
$\operatorname{supp}(f)$	Support of the function f

Vectors and Matrices

In general small roman letters refer to scalars or vectors. Uppercase roman letters are matrices. The respective dimensions are clear from the context. Vectors are column vectors.

$\mathrm{vec}(A)$	Column-wise reshaping the matrix A into a vector
$\mathrm{diag}(v)$	Diagonal matrix with diagonal entries v
$\otimes$	Tensor product
I_d	Identity matrix of size $d \times d$
$0_{d,d}$	Zero-matrix of size $d \times d$
$\mathbf{1}$	One-vector, dimension will be clear from the context
$\mathbf{0}$	Zero-vector, dimension will be clear from the context
$\det(A)$	Determinant of matrix A
$\ker(A)$	Kernel of matrix A
$\mathrm{rank}(A)$	Rank of matrix A
$\lVert \cdot \rVert_F$	Frobenius norm, $\lVert \cdot \rVert_F = \lVert \mathrm{vec}(\,\cdot\,) \rVert_2$

Figures

We mainly consider gray-value images, i.e., matrices of size $M \times N$. The smallest entry is set to the first color of the color map and the largest entry to the last color map. The remaining entries are scaled linearly.

In this thesis we mainly use two color maps: An inverted gray map where zero is set to white and the largest entry to black. Values in between appear in different shades of gray. The second color map uses red instead of black. Additionally blue is used if also negative entries appear. The range of the colors are then chosen symmetrically around zero to keep zero white.