Efficient Low-Rank Solution of Large-Scale Matrix Equations

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PUBLICATIONS

Large parts of this thesis have been published in the papers summarized below. The Chapters 3 and 4 are rearranged, partly revised, and extended versions of

> [37]: Peter Benner, Patrick Kürschner, Jens Saak: An Improved Numerical Method for Balanced Truncation for Symmetric Second Order Systems, Mathematical and Computer Modelling of Dynamical Systems, 19(6), pp. 593–615, 2013.
>
> [38]: Peter Benner, Patrick Kürschner, Jens Saak: Efficient Handling of Complex Shift Parameters in the Low-Rank Cholesky Factor ADI Method, Numerical Algorithms, 62(2), pp. 225–251, 2012.
>
> [36]: Peter Benner, Patrick Kürschner, Jens Saak: A Reformulated Low-Rank ADI Iteration with Explicit Residual Factors, Proceedings of Applied Mathematics and Mechanics, 19(1), pp. 585–586, 2013.
>
> [32]: Peter Benner, Patrick Kürschner: Computing Real Low-Rank Solutions of Sylvester Equations by the Factored ADI Method, Computers & Mathematics with Applications 67(9), pp. 1656–1672, 2014.

Chapter 5 includes

> [39]: Peter Benner, Patrick Kürschner, Jens Saak: Self-Generating and Efficient Shift Parameters in ADI Methods for Large Lyapunov and Sylvester Equations, Electronic Transactions on Numerical Analysis, 43, pp. 142–162, 2014.

but several substantial extensions were added.

Chapter 6, especially Section 6.2, contains a modified version of

> [41]: Peter Benner, Patrick Kürschner, Jens Saak: Low-Rank Newton-ADI methods for Large Nonsymmetric Algebraic Riccati Equations, Journal of the Franklin Institute 353(5), pp. 1147—1167, 2016.

Chapter 7 contains selected material from [37] as well as

> [33]: Peter Benner, Patrick Kürschner, Jens Saak: A Goal-Oriented Dual LRCF-ADI for Balanced Truncation, I. Troch and F. Breitenecker, eds., Vol. 7 of *Proceedings of the MathMod 2012, IFAC-PapersOnline, Mathematical Modelling*, Vienna, Austria, 2012, pp. 752–757.

> [178]: C. Nowakowski, Patrick Kürschner, Peter Eberhard, Peter Benner: Model Reduction of an Elastic Crankshaft for Elastic Multibody Simulations, ZAMM - Journal of Applied Mathematics and Mechanics 93(4), pp. 198–216, 2013.

> [40]: Peter Benner, Patrick Kürschner, Jens Saak: Frequency-Limited Balanced Truncation with Low-Rank Approximations, SIAM Journal on Scientific Computing 38(1), pp. A471–A499, 2016.

In particular, Section 7.3 is a slightly altered version of [40].

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ABSTRACT

In this thesis, we investigate the numerical solution of large-scale, algebraic matrix equations. The focus lies on numerical methods based on the alternating directions implicit (ADI) iteration, which can be formulated to compute approximate solutions of matrix equations in form of low-rank factorizations. These low-rank versions of the ADI iteration can be used to deal with large-scale Lyapunov and Sylvester equations. The major part of this thesis is devoted to improving the performance of these iterative methods. At first, we develop algorithmic enhancements that aim at reducing the computational effort of certain stages in each iteration step. This includes novel low-rank expressions of the residual matrix, which allows an efficient computation of the residual norm, and approaches for the reduction of the amount of occurring complex arithmetic operations. ADI based methods rely on shift parameters, which influence how fast the iteration generates an approximate solution. For this, we propose novel shift generation strategies which improve the convergence speed of the low-rank ADI iteration and, at the same time, can be performed in an automatic and cost efficient numerical way. Later on, the improved low-rank ADI methods for Lyapunov and Sylvester equations are used in Newton type methods for finding approximate solutions of quadratic matrix equations in the form of symmetric, continuous-time, but also more general nonsymmetric, algebraic Riccati equations. In the last part of this thesis, the methods for solving large-scale Lyapunov equations are applied in order to execute balanced truncation model order reduction for linear control systems in a numerically feasible way. For frequency-limited balanced truncation, which is a special variant of balanced truncation, a novel algorithmic framework is proposed that enables an efficient numerical execution of this model reduction technique.

ZUSAMMENFASSUNG

Die vorliegende Arbeit befasst sich mit der numerischen Lösung von großskaligen, algebraischen Matrixgleichungen. Der Fokus liegt hierbei auf numerischen Verfahren, die auf der Methode der alternierenden Richtungen (ADI) basieren und so formuliert werden können, dass sie Näherungslösungen von Matrixgleichungen in Form von Niedrigrangfaktorisierungen berechnen. Diese Niedrigrang-Versionen der ADI Iteration sind in der Lage, hochdimensionale Lyapunov- und Sylvestergleichungen zu behandeln. Im Hauptteil dieser Arbeit werden numerische Verbesserungen dieser iterativen Verfahren untersucht. Dazu werden einige algorithmische Erweiterungen entwickelt, um den Rechenaufwand von bestimmten Abschnitten der einzelnen Iterationsschritte zu senken. Dies beinhaltet neue Niedrigrangdarstellungen des Residuums, welche eine receneffiziente Berechnung der Residuumsnorm erlauben, sowie Strategien zur Verringerung der Anzahl auftretender Rechenoperationen in komplexer Arithmetik. Die ADI Iteration benötigt Shiftparameter, die beeinflussen, wie schnell das Verfahren eine Näherungslösung findet. Neuartige Strategien zur Shiftberechnung werden vorgestellt, die sowohl die Konvergenzgeschwindigkeit der Niedrigrang-ADI Iteration verbessern, als auch automatisch und kosteneffizient durchgeführt werden können. Die verbesserten Niedrigrang-ADI Iterationen werden später innerhalb Newton-artiger Verfahren angewendet, um Näherungslösungen von quadratischen Matrixgleichungen zu berechnen. Dabei werden symmetrische, zeit-kontinuierliche und auch die allgemeineren unsymmetrischen, algebraischen Riccatigleichungen betrachtet. Im letzten Abschnitt der vorliegenden Arbeit finden die Verfahren zur numerischen Lösung von großskaligen Lyapunovgleichungen Anwendung, um eine Modellordnungsreduktion für lineare Regelungssysteme mittels balancierten Abschneidens mit geringen Rechenaufwand durchzuführen. Für balanciertes Abschneiden in begrenzten Frequenzintervallen, eine Sonderform des balancierten Abschneidens, wird eine neue numerische Strategie erarbeitet, die eine effiziente numerische Durchführung dieses speziellen Modellordnungsreduktionsverfahrens ermöglicht.

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