

WEB-Spline Approximation and Collocation for Singular and Time-Dependent Problems

Von der Fakultät Mathematik und Physik der Universität Stuttgart
zur Erlangung der Würde eines
Doktors der Naturwissenschaften (Dr. rer. nat.)
genehmigte Abhandlung

vorgelegt von

Florian Martin

aus Bietigheim-Bissingen, Deutschland.

Hauptberichter:	Prof. Dr. Klaus Höllig
Mitberichter;	Prof. Dr. Guido Schneider
2. Mitberichter:	Prof. Dr. Ulrich Reif

Tag der Einreichung:	06.04.2017
Tag der mündlichen Prüfung:	06.07.2017

Institut für Mathematische Methoden in den Ingenieurwissenschaften,
Numerik und Geometrische Modellierung der Universität Stuttgart

2017

Berichte aus der Mathematik

Florian Martin

**WEB-Spline Approximation and Collocation for
Singular and Time-Dependent Problems**

D 93 (Diss. Universität Stuttgart)

Shaker Verlag
Aachen 2017

Bibliographic information published by the Deutsche Nationalbibliothek

The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at <http://dnb.d-nb.de>.

Zugl.: Stuttgart, Univ., Diss., 2017

Copyright Shaker Verlag 2017

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior permission of the publishers.

Printed in Germany.

ISBN 978-3-8440-5428-6

ISSN 0945-0882

Shaker Verlag GmbH • P.O. BOX 101818 • D-52018 Aachen

Phone: 0049/2407/9596-0 • Telefax: 0049/2407/9596-9

Internet: www.shaker.de • e-mail: info@shaker.de

Abstract

The geometric shape of construction components can cause difficulties for a straightforward application of finite element simulations. In particular, for the computation of deformations with linear elasticity models, edges and vertices often lead to singularities which significantly affect the accuracy of approximations. To reduce the error, adaptive algorithms with automatic grid refinement in critical regions are essential and present powerful tools in the numerical solution of partial differential equations.

In view of the regular grid structure, B-splines are ideally suited for hierarchical refinement via adaptive algorithms. Moreover, weight functions can model complicated geometries without having to resort to time-consuming mesh generation. Finally, contrary to classical finite element bases, the pointwise residual of the partial differential equations can be computed for B-spline approximations. This leads to novel refinement strategies which are not only more natural but also much easier to implement.

In this thesis, adaptive refinement algorithms are developed and analyzed for the web-method, one of the two most popular finite element discretizations with B-splines. The analysis is built upon the extensive theory available for uniform spline approximations, hierarchical techniques with classical C^0 -elements, and some ideas from the earlier work on Poisson's equation as an elementary test case. Furthermore, the Lamé-Navier equations of linear elasticity serve as a principal model problem, noting that the new methods apply to simpler elliptic problems as well.

The advantages of B-splines play a crucial role for web-collocation, which is considered as second numerical approximation method in this thesis. The recently introduced uniform collocation algorithm for web-splines is generalized to hierarchical spline spaces under usage of a newly developed reordering algorithm. Without affecting the accuracy, it prevents coincident collocation points in the transition area

of subsequent hierarchical levels, naturally occurring in case of multi-level Greville abscissae. As for finite element schemes, the development of hierarchical refinement techniques is essential for problems with singularities and leads to substantial improvements in accuracy.

In addition to elliptic equations and systems, time-dependent problems, including the heat equation and wave equation as simplest examples, are considered. Following a common approach, stationary solution techniques are combined with various time-step procedures. The resulting novel techniques eliminate the time-consuming mesh generation process and, for collocation, do not require numerical integration. Their performance is much superior to classical approximations and is illustrated for the modeling of tsunamis in the thesis.

The application of hierarchical spline spaces in both web-methods is facilitated by the exploration of a general concept of extending B-splines, involving basis functions with different grid widths. This is motivated by a counterexample which shows the failure of the polynomial precision for the approach of transferring the uniform procedure straightforward.

The benefits of hierarchical splines are not only explored numerically for the described examples but also in theory. It is shown that hierarchical spline spaces have the potential of achieving a specified level of approximation accuracy with minimal dimension. By considering a typical class of singular functions, this result generalizes the corresponding conclusion for uniform spaces regarding smooth functions. Furthermore, comparisons between the theoretical and numerical results confirm the optimality of the spaces generated by both adaptive approximation schemes.

Kurzzusammenfassung

Die Anwendung eines Finite-Elemente-Verfahrens auf die meisten Bauteile, die in der heutigen Zeit verwendet werden, wird durch ihre Form erschwert. Insbesondere bei der Berechnung der Auslenkung unter Belastung in der linearen Elastizitätstheorie, können Ecken und Kanten zu Singularitäten mit erheblicher Auswirkung auf die Genauigkeit der numerischen Approximation führen. Aus diesem Grund ist die Anwendung eines adaptiven Algorithmus, der diese Problemstellen automatisch erkennt und an diesen eine Gitterverfeinerung durchführt, unverzichtbar für die Verringerung des Fehlers geworden.

Aufgrund der regulären Gitterstruktur sind B-Splines ideal für hierarchische Verfeinerungen, wie sie im Laufe eines adaptiven Verfahrens auftreten, geeignet. Außerdem erlaubt die Multiplikation mit einer Gewichtsfunktion die Modellierung komplizierter Geometrien und vermeidet dadurch eine aufwendige Vernetzung. Desweiteren ermöglichen B-Splines, im Gegensatz zu den klassischen Basisfunktionen, das punktweise Auswerten des Residuums der betrachteten Differentialgleichung und damit die Entwicklung neuer Verfeinerungsstrategien, die einerseits einfacher zu implementieren und andererseits natürlicher als die herkömmlichen sind.

Diese Arbeit beinhaltet die Entwicklung und Analyse von adaptiven Verfeinerungsalgorithmen für die WEB-Methode, welche eine der zwei bekanntesten Finite-Elemente-Verfahren mit B-Splines darstellt. Hierzu werden die bereits existierende umfangreiche Forschung für die Approximation mit uniformen Splines, die hierarchischen Verfeinerungstechniken für klassische C^0 -Elemente und die Erkenntnisse aus der Untersuchung der Poisson-Gleichung als elementares Beispiel verwendet. Die Lamé-Navier Gleichungen der linearen Elastizität dienen darüber hinaus als Modellproblem, wobei die neu entwickelten Methoden auch auf einfachere elliptische Randwertprobleme angewendet werden können.

Die Vorteile von B-Splines spielen auch bei dem zweiten in dieser Arbeit betra-

chteten numerischen Approximationsverfahren, der WEB-Kollokation, eine wichtige Rolle. Unter Verwendung eines neu entwickelten Algorithmus zur Umsortierung von Kollokationspunkten wird der uniforme Kollokationsalgorithmus für WEB-Splines auf hierarchische Spline-Räume verallgemeinert. Dabei werden zusammenfallende Kollokationspunkte in den Übergangsbereichen von aufeinanderfolgenden hierarchischen Leveln, wie sie bei der Verwendung der Greville Abszissen auftreten, verhindert ohne die Genauigkeit zu beeinträchtigen. Die ebenfalls durchgeführte Entwicklung von hierarchischen Verfeinerungstechniken ist auch bei dieser Methode notwendig um eine erhebliche Verbesserung der Genauigkeit bei Problemen mit Singularitäten zu erzielen.

Zusätzlich zu elliptischen Gleichungen behandelt diese Arbeit auch zeitabhängige Probleme, mit der Wärmeleitungs- und Wellengleichung als repräsentative Beispiele. Einem konventionellen Lösungsansatz folgend, werden Lösungsmethoden für stationäre Probleme mit verschiedenen Zeitschrittverfahren kombiniert. Die resultierenden neuen Lösungsverfahren vermeiden eine zeitaufwendige Gittererzeugung, sowie im Falle der Kollokation eine numerische Integration. Hieraus ergeben sich klare Vorteile gegenüber klassischen Approximationsmethoden, die anhand der Modellierung eines Tsunami demonstriert werden.

Der Einsatz von hierarchischen Spline-Räumen in beiden WEB-Methoden wird durch die Einführung einer allgemeineren Erweiterung von B-Splines ermöglicht, welche auf Basisfunktionen mit unterschiedlichen Gitterweiten anwendbar ist. Die Notwendigkeit einer solchen Verallgemeinerung ist durch ein Gegenbeispiel motiviert, welches das Fehlschlagen der exakten Darstellung von Polynomen bei einer intuitiven Anwendung der uniformen Erweiterung aufzeigt.

Neben den numerischen Vorzügen von hierarchischen Splines, die sich aus den durchgeführten Berechnungen ergeben, wird außerdem die Eigenschaft der optimalen Approximation von glatten Funktionen durch uniforme Splines mit Hilfe von hierarchische Räume auf eine typische Klasse singulären Funktionen verallgemeinert. Ein Vergleich zwischen diesen theoretischen und numerischen Resultaten zeigt zudem, dass die entwickelten adaptiven WEB-Verfahren hierarchische Spline-Räume mit optimaler Approximationseigenschaft erzeugen.

Acknowledgment

I would like to sincerely thank Professor Klaus Höllig for the possibility of working on this thesis and contributing to the improvement of the various web-methods. In addition, I want to express my gratitude to the employees of the former chair for Numerical Analysis and Computer-Aided Geometric Design at the University of Stuttgart, Germany, for supporting me in different aspects of this thesis and their cooperation in additional projects. A special thanks goes to Professor Guido Schneider (University of Stuttgart) and Professor Ulrich Reif (Technical University of Darmstadt) for the support in special topics of this thesis. My final thank goes to Tobias and Nora for the additional proofreading.

Contents

List of Figures	XI
List of Tables	XVI
Notation and Symbols	XVII
1 Introduction	1
1.1 Outline of the Thesis	4
1.2 Notation	5
2 WEB-Splines	7
2.1 B-Splines	7
2.2 Weight Functions	12
2.2.1 Construction of Weight Functions	13
2.3 WEB-Splines	16
2.3.1 Marsden's Identity	17
2.3.2 Extension of B-Splines	18
2.4 Hierarchical WEB-Splines	21
2.4.1 Hierarchical Splines	21
2.4.2 Extension of a Hierarchical B-Spline Basis	26
2.4.3 Overview	34
2.5 Dual Functions	34
3 Hierarchical Quasi-Interpolant	37
3.1 Polynomial Approximation	38
3.2 Quasi-Interpolants	39
3.2.1 Example 1: Quasi-Interpolant for L^2 -Functions	40

3.2.2	Example 2: Schoenberg's Scheme	41
3.3	Hierarchical Quasi-Interpolant	43
3.3.1	Projection	46
3.3.2	Boundedness	47
3.4	Error Estimation	50
3.5	Asymptotic Optimal Grids	52
3.5.1	$\Omega = [0, 1]$ and $p = \infty$	52
3.5.2	$\Omega = [0, 1]$ and $p = 2$	59
3.5.3	$\Omega = [0, 1]^2$ and $p = 2$	62
3.5.4	L-Shaped Domain for $p = 2$	67
4	Adaptive Finite Element Method with Weighted B-Splines	69
4.1	Basic Principles	70
4.1.1	Abstract Variational Problem	70
4.1.2	Finite Element Method with Weighted Splines	74
4.2	A-Posteriori Error Estimation	77
4.3	Adaptive Algorithm	85
5	Adaptive WEB-Collocation	91
5.1	Uniform WEB-Collocation	92
5.2	Hierarchical WEB-Collocation	95
5.3	Adaptive Collocation Algorithm	100
6	Elliptic Problems with Singular Solutions	105
6.1	Laplace Equation	106
6.1.1	Numerical Results	108
6.2	Linear Elasticity	117
6.2.1	Lamé-Navier Equations	117
6.2.2	Plane Strain and Plane Stress	120
6.2.3	Singular Solutions	123
6.2.4	Numerical Approximation with B-Splines	127
6.2.5	Numerical Results	128
6.3	Conclusion	134
7	Time-Dependent Problems with WEB-Collocation	137
7.1	Heat Equation	138

7.1.1	Basic Model	138
7.1.2	Example	141
7.2	Tsunami Modeling with the Wave Equation	144
7.2.1	Shallow Water Equations	144
7.2.2	Approximation	148
7.2.3	Example	149
8	Summary	155
	Bibliography	159

List of Figures

1.1	Calculation domains leading to singularities in the solutions.	2
a	"Battery Problem" with different materials.	2
b	Reentrant corner with angle α at the boundary.	2
2.1	The B-splines b^0, b^1 and b^2 and their derivatives.	9
a	B-spline b^0	9
b	B-spline b^1	9
c	B-spline b^2	9
2.2	Cubic B-splines with grid width $h = 1/2$	10
2.3	The multivariate B-spline $b_{(0,0),1/2}^3$ with the grid structure of its support.	11
a	B-spline $b_{(0,0),1/2}^3$	11
b	Grid structure of $\text{supp } b_{(0,0),1/2}^3$	11
2.4	Possible intersections of B-splines with a circle shaped domain.	12
2.5	A weight function for a disc.	14
a	Weight function.	14
b	Resulting domain.	14
2.6	A weight function for the outside of a circle.	15
a	Weight function.	15
b	Resulting domain.	15
2.7	L-shaped domain with a reentrant corner.	17
2.8	Classification of a set of relevant biquadratic B-splines b_k^2	19
a	Classification based on the grid cells.	19
b	Classification based on the midpoints of the support.	19
2.9	Hierarchical biquadratic spline space with three levels.	23
2.10	Subdivision of a quadratic B-spline.	24
2.11	Initial situation for the transformation of a non strongly nested grid.	25

2.12	Violation of the strongly nested condition.	26
2.13	Transformed grid satisfying the strongly nested condition.	27
2.14	Hierarchical spline space for the counterexample.	28
2.15	Dual function λ_0 for the linear B-spline b_0^1	35
3.1	Error of Schoenberg's scheme for $f(x) = \sqrt{x}$	42
3.2	Nonuniform grid for Schoenberg's scheme.	43
3.3	The composition of the support of P_1^1	48
3.4	Construction of a hierarchical grid with boundary points r_k	55
3.5	The determination of the distance of a grid cell $D_{(l,i)}$ to the origin.	64
3.6	Construction of a 2D hierarchical grid with boundary points.	66
3.7	Grid construction for a L-shaped domain.	68
4.1	Lipschitz domain with coverage of a part of its boundary.	71
4.2	Three-quarter circle with a reentrant corner at $(1/2, 1/2)$	76
4.3	Singular solution u of Poisson's equation on a three-quarter circle.	77
a	Solution u	77
b	2-norm of ∇u	77
4.4	Error of an approximation of a singular solution of Poisson's equation.	78
a	Absolute value of the error	78
b	Distribution of the error on the grid cells.	78
4.5	The different sections of $\partial(D_{(l,i)} \cap \Omega)$	82
4.6	Value of the residual based error estimator for Poisson's equation.	85
4.7	Evolution of the grid in the course of an adaptive algorithm.	90
a	Uniform starting grid.	90
b	One refinement level.	90
c	Two refinement levels.	90
5.1	Classification of B-splines for the web-collocation method.	94
5.2	Determination of a boundary collocation point by bisection.	95
5.3	Location of all collocation points for a three-quarter circle.	96
5.4	Reorder algorithm for a hierarchical grid and quadratic B-splines.	97
5.5	Reorder algorithm for a hierarchical grid and cubic B-splines.	98
5.6	Reorder algorithm for a hierarchical grid and quartic B-splines.	98
5.7	Reorder algorithm for a hierarchical grid and quintic B-splines.	98
5.8	Problematic situation for overlapping boundary collocation points.	99

5.9	Approximation of a singular solution with a hierarchical cubic spline.	102
a	Solution and approximation.	102
b	Grid function and collocation points.	102
5.10	Approximation of a singular solution with a hierarchical cubic spline.	103
a	Development of the residual for certain iteration steps.	103
b	The error in the L^2 -norm in dependence on the dimension.	103
5.11	Ratio between the dimension and the exponentiated error.	104
6.1	Domain with a reentrant corner with angle α .	106
a	Choice of the polar coordinates.	106
b	Relation of the angle to the x_1 -axis.	106
6.2	Example solution u of the Laplace equation.	108
a	Solution u .	108
b	Partial derivative $\partial_r u$.	108
6.3	The domains for the adaptive algorithms.	109
a	Finite element method.	109
b	Collocation method.	109
6.4	The relative L^2 -error of the adaptive finite element approximation.	112
a	Relative L^2 -error for $n = 2$.	112
b	Relative L^2 -error for $n = 3$.	112
c	Relative L^2 -error for $n = 4$.	112
d	Relative L^2 -error for $n = 5$.	112
6.5	The relative H^1 -error of the adaptive finite element approximation.	113
a	Relative H^1 -error for $n = 2$.	113
b	Relative H^1 -error for $n = 3$.	113
c	Relative H^1 -error for $n = 4$.	113
d	Relative H^1 -error for $n = 5$.	113
6.6	The relative error of the adaptive collocation approximation.	114
a	Relative L^2 -error for $n = 2$.	114
b	Relative L^2 -error for $n = 3$.	114
c	Relative L^2 -error for $n = 4$.	114
d	Relative L^2 -error for $n = 5$.	114
6.7	Comparison of the computation time for both methods.	116
a	Computation times for $n = 2$.	116
b	Computation times for $n = 3$.	116

c	Computation times for $n = 4$.	116
d	Computation times for $n = 5$.	116
6.8	Cross section of a bridge with acting forces.	117
6.9	Displacement of a bridge under the gravitational force.	122
6.10	Displacement of a rotating disc.	123
6.11	Example domain for a singular solution of the linear elasticity.	124
6.12	The function for the determination of the singularity parameter.	127
a	Mode 1 solution.	127
b	Mode 2 solution.	127
6.13	Mode 2 solution u of the elasticity problem.	129
a	First component of u .	129
b	Second component of u .	129
6.14	The relative error of the adaptive finite element approximation.	130
a	Relative L^2 -error for $n = 2$.	130
b	Relative H^1 -error for $n = 2$.	130
6.15	The relative error of the adaptive finite element approximation.	131
a	Relative L^2 -error for $n = 3$.	131
b	Relative H^1 -error for $n = 3$.	131
c	Relative L^2 -error for $n = 4$.	131
d	Relative H^1 -error for $n = 4$.	131
e	Relative L^2 -error for $n = 5$.	131
f	Relative H^1 -error for $n = 5$.	131
6.16	The relative error of the adaptive collocation approximation.	133
a	Relative L^2 -error for $n = 2$.	133
b	Relative L^2 -error for $n = 3$.	133
c	Relative L^2 -error for $n = 4$.	133
d	Relative L^2 -error for $n = 5$.	133
6.17	Comparison of the computation time for both methods.	134
a	Computation times for $n = 3$.	134
b	Computation times for $n = 5$.	134
6.18	The ratio between the computation time and the dimension.	135
a	$n = 2$.	135
b	$n = 3$.	135
c	$n = 4$.	135

d	$n = 5$	135
7.1	Example solution u of the heat equation.	141
a	Solution u for $t = 0$.	141
b	Solution u for $t = 0.1$.	141
7.2	Example solution of the heat equation for a fixed $x_2 = 0$.	142
7.3	The error for the approximation of the solution of the heat equation.	143
a	Error for $n = 2$.	143
b	Error for $n = 5$.	143
7.4	General setup of a two-dimensional water flow.	144
7.5	Simple map of the gulf of Mexico.	149
7.6	Scattered bathymetric data for the gulf of Mexico.	150
7.7	GUI for a simple simulation of a tsunami in the gulf of Mexico.	151
7.8	Simulation of a tsunami in the gulf of Mexico.	152
7.9	Simulation of a tsunami in the gulf of Mexico (continued).	153

List of Tables

2.1	R-functions for standard set operations.	15
2.2	Rounded values for the general extension coefficients.	34
6.1	Number of marked cells for an adaptive finite element approximation.	113
6.2	Number of marked cells for different parameters.	132
7.1	Estimated convergence rate for the derived approximation scheme. . .	143

Notation and Symbols

General

$\mathbb{N}_0, \mathbb{N}$	natural numbers with and without zero
$\mathbb{Z}, \mathbb{R}, \mathbb{C}$	integers, real and complex numbers
$\ \cdot\ _{l,\Omega}$	Sobolev norm on the space $W^{l,2}(\Omega) = H^l(\Omega)$
$\preceq, \succeq, \asymp$	(in)equalities up to a constant
$C(\cdot)$	constant with dependency on parameters
$\partial_\nu, \partial_{x_\nu}, \partial^\alpha$	partial derivatives
diam	diameter of a set
dist	distance function
supp	support of a function

B-Splines

b^n	uniform B-spline of degree n
$b_{k,h}^n, b_k^n, b_k$	scaled translated tensor product B-spline
B_i	web-spline
$\mathbb{B}_h^n(\Omega)$	spline space on a bounded domain
$\Xi_h^n(\Omega)$	hierarchical spline space on a bounded domain
(l, i)	node in a hierarchical spline tree
$w\mathbb{B}_h^n$	weighted spline space
$w^e\mathbb{B}_h^n(\Omega)$	web-space
D_k^l	support of a B-spline in a hierarchical basis
$e_{i,j}$	extension coefficients
$I, I_{(l,i)}, J, J_{(l,i)}, K, K_{(l,i)}$	set of inner, outer and relevant B-spline indices

h, h_l	grid width
λ_k, λ_k^l	dual function of a B-spline b_{k, h_l}
$s(l)$	number of nodes on level l in a hierarchical tree
$\xi_{(l,i)}$	defining cube of a node in a hierarchical spline space
w	weight function

Quasi-Interpolants

$D_{(l,i)}$	relevant grid cell for a hierarchical spline space
Q, Q^l	uniform quasi-interpolant
Q_k, Q_k^l	linear functionals of a uniform quasi-interpolant
P^L	hierarchical quasi-interpolant with L levels
P_k^l	linear functionals of a hierarchical quasi-interpolant
P_n	projection on polynomials of degree n

Finite Elements and Collocation

a	elliptic, symmetric bilinear form on a Hilbert space H
Δ	Laplace operator
$\partial^\perp$	normal derivative
ε	refinement parameter for an adaptive collocation algorithm
$\eta_R, \eta_{D_{(l,i)}, R}$	residual based error estimator
H_Γ^1	constrained Sobolev space
$L^2(\Omega)$	square integrable functions
λ	linear bounded functional on a Hilbert space H
∇	gradient
$\Omega \subset \mathbb{R}^d$	open, bounded, Lipschitz domain
$\partial\Omega$	domain boundary
$R(u_h), r(u_h)$	inner and boundary residual of an approximation u_h
$u_h \approx u$	finite-dimensional approximation of a solution u
θ	refinement parameter for an adaptive finite element algorithm
ξ	outward normal vector
ξ_m	collocation point

Elliptic and Time-Dependent Problems

Δt	time step $[s]$
div	divergence operator
E, ν	Young's modulus $[\frac{N}{m^2}]$ and Poisson number $[-]$
$\eta(x_1, x_2, t)$	water elevation above zero $[m]$
$\varepsilon(u), \sigma(u)$	strain and stress tensor
g	gravitational acceleration $[\frac{m}{s^2}]$
$h(x_1, x_2)$	water depth $[m]$
λ, μ	Lamé constants $[-]$